πEQUALITY, EXPRESSIONS, EQUATIONS, AND INEQUALITIES

An equation is a mathematical statement indicating the equivalence of two quantities or expressions. An inequality is a mathematical statement indicating the relative magnitudes of two quantities or expressions that are not equivalent.

STRAND 1. Relationships (Generalizing Relationships)
This strand identifies concepts related to relationships expressed in their generalized form* as expressions, equations, or inequalities.

*Some of the concepts might also apply to relationships between numerical quantities that do not have a generalized form (e.g., 3 < 4) and that might be examined by students as a bridge into generalized forms.

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1.1) Two quantities will relate in one of three ways: (1) the quantities will be equal; (2) one quantity will be greater than the other quantity; (3) one quantity will be less than the other quantity.	• Understand that two quantities can relate in one of three ways: they can be equal, one quantity can be larger than the other, or one quantity can be smaller than the other (Trichotomy Property)	• Statements showing appropriate use of =, < and > to describe the relative magnitudes of two quantities			• Collect 50 cents (Fosnot & Jacob, 2009, grade 2) • Trichotomy Prop (Blanton et al., 2011, grades 3-5)
1.2) Equations depict relationships for which one of the following holds: • The equation or system of equations is true for all values of the variable(s) (i.e., mathematical	• Understand that mathematical identities and properties are equations that are true for all values of the variable • Understand when an equation is true for all values of the variable(s)	• Statements indicating operations on equations that reflect appropriate understanding of numbers of values of variable(s) for which equation is true • Statements explaining why equation is true for	• Students will sometimes try to "solve" identities • Students often view the equal sign operationally (Alibali, 1999; Baroody & Ginsburg, 1983; Behr et al., 1980; Carpenter et al., 2003; Falkner et al., 1999; Kieran, 1981;		• $2x + 3 = 5x – 9$ $2(3x + 4) = 6x + 8$ $2(3x + 4) = 6x – 5$ (Huntley et al., 2007, grade 11)

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identities or properties; 5x – 3x = 3x - x). • The equation is true for some finite (nonempty) set of values of the variable(s) (e.g., functions; 5x + 2 = 12). • The equation is not true for any values of the variable (e.g., 2x – x = x + 3).	• Understand when an equation is true for one or some but not all values of the variables • Understand when an equation has no solution	particular number of values of the variable(s) (all, some, one, or none) (e.g., justifications for why equation such as a + 5 = 10 has one solution and a – a = 0 has infinitely many solutions) • Statements about whether the equation is to be solved or whether the equation describes a general principle • Statements indicating that equations with multiple variables may have more than one solution	Knuth et al., 2005, Knuth et al., 2006; Knuth et al., 2008; McNeil & Alibali, 2004, 2005; Morris, 2003) across grades 1-8		
1.3) A system of linear equations depicts a relationship between two co-varying quantities for which one of the following holds: • The system has infinitely many solutions • The system has a single solution	• Understand that a system of linear equations could have no, one, or infinitely many solutions • Understand the relationship between the graph of a system of linear equations and the number of solutions the system has	• Statements indicating the number of solutions a system has, given a system of equations • Statements indicating the number of solutions a system has, given the graph of a system			

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• The system has no solution					
1.4) Inequalities depict different types of relationships for which one of the following holds: • The inequality is true for all values of the variable(s) (e.g., $2x + 6 \leq 2(x + 3)$). • The inequality is true for one or some values of the variable(s) (e.g., $5x - 3 < 0$). • The inequality is not true for any values of the variable (e.g., $4x - 2x > 3 + 2x$).	• Understand when an inequality is true for all values of the variable(s) • Understand when an inequality is true for one or some solutions, but not all possible values of the variable • Understand when an inequality has no solution	• Statement indicating the number of solutions an inequality has			
1.5) Properties of Equations govern the transformation of equations and include: • Addition Property: If $a = b$, then $a + c = b + c$. • Subtraction Property: If $a = b$, then $a - c = b - c$.	• Understand why adding the same quantity or expression to both sides of an equation preserves the equivalence • Understand why subtracting the same quantity or expression from both sides of an equation preserves the	• Written mathematical work illustrating performing the same operation on both sides of an equation with an explanation indicating why this procedure makes sense			

DO NOT USE WITHOUT PERMISSION

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• Multiplication Property: If a = b, then a × c = b × c. • Division Property: If a = b and c ≠ 0, then a ÷ c = b ÷ c.	equivalence • Understand why multiplying both sides of an equation by the same quantity preserves the equivalence • Understand why dividing both sides of an equation by the same nonzero quantity preserves the equivalence				
1.6) Properties of Inequalities govern the transformation of inequalities and include: • Addition Property: If a < b, then a + c < b + c (similarly for >, ≤, ≥ and ≠). • Subtraction Property: If a = b, then a – c < b – c (similarly for >, ≤, ≥ and ≠). (This intentionally does not include multiplication or division because it is beyond grades 3-8.)	• Understand why adding the same quantity or expression to both sides of an inequality preserves the inequality relationship • Understand why subtracting the same quantity or expression from both sides of an inequality preserves the inequality relationship	• Operations on linear inequalities that demonstrate understanding that adding or subtracting a quantity from both sides of the inequality preserves the relationship			
1.7) The symbol '='	• Understand why a	• Mathematical statements	• Students often view		

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expresses reflexive, symmetric, and transitive relationships.	quantity or expression is equal to itself (a=a) • Understand why for quantities a and b, if a = b, then b = a • Understand why for quantities a, b and c, if a = b and b = c, then a = c	that depict reflexive, symmetric or transitive property of equality • Mathematical statements illustrating application of reflexive, symmetric, or transitive properties	the equal sign operationally (Alibali, 1999; Baroody & Ginsburg, 1983; Behr et al., 1980; Carpenter et al., 2003; Falkner et al., 1999; Kieran, 1981; Knuth et al., 2005, Knuth et al., 2006; Knuth et al., 2008; McNeil & Alibali, 2004, 2005; Morris, 2003) across grades 1-8		
1.8) Properties of Inequality Signs: The symbols <, ≤, >, ≥ express transitive relationships. (This intentionally does not include the other related properties, e.g., reflexive, because this is beyond grades 3-8.)	• Understand why if a < b and b < c, then a <c (similarly for >, ≤, ≥)				

STRAND 2. Representations (Representing Generalizations)
Generalized relationships can be represented in a variety of forms, including words, symbols, tables and graphs. This strand addresses representational knowledge for modeling, describing, or reasoning with expressions, equations and inequalities.

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2.1) The equal sign is used to represent the equivalence of two quantities or mathematical expressions.	• Understand the equal sign as a relational (rather than operational) symbol . • Interpret meaning of equations represented in symbolic form (e.g., understand that a = a represents that the measure of a quantity is equal to itself)	• Correct responses to "true/false" and "open" equations that support a relational view of the equal sign • Written or oral statements that indicate the equal sign means "the same as"	• Students often view the equal sign operationally (Alibali, 1999; Baroody & Ginsburg, 1983; Behr et al., 1980; Carpenter et al., 2003; Falkner et al., 1999; Kieran, 1981; Knuth et al., 2005, Knuth et al., 2006; Knuth et al., 2008; McNeil & Alibali, 2004, 2005; Morris, 2003) across grades 1-8	• Solve T/F equations (Lee & Wheeler, 1989, grade 10) • Solve T/F and open number sentences (Carpenter et al., 2003, grades 1-6)	• T/F sentences (Carpenter et al., 2003, grades 1-6) • Open number sentences (Carpenter et al., 2003, grades 1-6) • Equal sign definition (Knuth et al., 2005; Knuth et al., 2006, grades 6-8) • Collect 50 cents (Fosnot & Jacob, 2009, grade 2)
2.2) The symbols <, >, ≤, ≥ and ≠ are used to represent the relative magnitude of two quantities or expressions.	• Understand the meaning of the symbols < and > and how they are used • Interpret meaning of inequality relationships represented in symbolic form (e.g., understand that a ≤ a represents that the measure of a quantity is less than or equal to	• Mathematical statements that show correct use of symbols < and >			

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	itself)				
2.3) Expressions, equations, or inequalities can be used to represent mathematical situations.	• Understand how to use equations to express arithmetic properties • Understand how to write and interpret expressions that represent situations involving unknown, varying quantities • Understand how to write and interpret simple (one-step) equations such as a + c = b, a – c = b a × c = b, a ÷ c = b, etc, to describe situations • Understand how to write and interpret multi-step equations (e.g., of the form ax + b = c, ax + b = cx + d) to represent problem situations • Understand how to write and interpret simple inequalities such as a + c < b, a – c < b a × c < b, a ÷ c < b to describe situations (similarly for >, ≤, ≥, ≠) • Understand how to write	• Equations accurately expressing arithmetic properties • Expressions representing unknown, varying quantities • Statements indicating acceptance of ambiguity of expressions by refraining from assigning a particular value to "some unknown amount" or attempting to "solve" an algebraic expression • Appropriate use of equations to model linear mathematical situations • Appropriate use of inequalities to model linear mathematical situations	• Students have difficulty generating equations to represent word problems • Students generally prefer to use arithmetic rather than equations to solve problems (Kieran, 2007) • Students have difficulty accepting that one can represent and reason with an algebraic expression (quantity) whose specific numerical value is not known (e.g., the number of pieces of candy in John's pocket) (Carraher & Schliemann, 2007, grades 2-4) • Students often assign numerical values to unknown quantities (Firth, 1975; cited in Kieran, 1989, grade 9; Blanton, 2008, grades 3-5; Carraher &	• Use equations to express arithmetic properties (Carpenter et al., 2003) • Write expressions to represent situations involving varying quantities (Carraher & Schliemann, 2007, grades 2-4; Carraher et al., 2008, grades 2-4) • Write simple number sentences such as a + x = b, a – x = b a × x = b, a ÷ x = b to describe situations (Carpenter et al., 2003, grades 1-6) • Write equations of the form ax + b = c to represent problem situations (Brenner et al., 1997, grades 7-8;	• Wallet task (Carraher et al., 2008,, grades 2-4) • Frog jumping (Fosnot & Jacob, 2009, grade 5) • Equation writing and solving (Swafford & Langrall, 2000, p. 93, grade 6) • Pet, Candy bar (Johanning, 2004, grades 6-8) • Stone piles (NRC, 2001, p. 257) • Triangle, Mark, Bus (Stacey & MacGregor, 2000, grades 9-10) • Equation writing and solving (Brenner et al., 1997, grades 7-8) • Cheapest club, Parking lots,

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	and interpret simple (one-step) and multi-step inequalities of the form (e.g., ax + b = c, ax + b < cx + d) to represent problem situations		Schliemann, 2007, grades 2-4) • Students sometimes think that expressions can be "solved"	Johanning, 2004, grades 6-8; Stacey & MacGregor, 2000, grades 7-10; Swafford & Langrall., 2000, grade 6) • Write equations of the form ax + b = cx + d to represent problems (Kieran & Sfard, 1999, grade 7; Stacey & MacGregor, 2000, grades 7-10; Yerushalmy, 2000, grades 7-9) • Use variables (letters) to represent an unknown number, and write "function like" algebraic expressions with this variable (e.g., 3 more than N is N + 3) (Carraher et al., 2006, grades 2-4)	Bonus (Yerushalmy, 2000, grades 7-9) • Number (Stacey & MacGregor, 2000, grades 9-10) • Best Deal (Brizuela, 2003, grades 2-3) • Frog jumping (Fosnot & Jacob, 2009, grade 5)
2.4) Equations in one	• Understand why, in an	• Solutions to equations of			

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variable can be interpreted as representing where two functions are equal (e.g., 2x + 1 = 3x – 2 is, functionally, where y=2x+1 and y=3x-2 intersect or are equal).	equation in one variable, each expression can be interpreted as corresponding to a function. • Understand why equating two corresponding functions can be interpreted as representing where two functions intersect or are equal.	the form ax + b = cx + d by graphing y = ax + b and y = cx + d and finding the point of intersection			

STRAND 3. Justifications (Justifying Generalizations)

Generalized relationships can be justified[1] using different strategies. This strand describes how justification occurs in the context of EEEI.

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3.1) Generalizations about properties of equations and equality are sometimes justified empirically, by substituting specific numbers in the generalization.	• Understand how to develop empirical arguments (e.g., testing examples) to examine conjectures about arithmetic relationships • Understand limitations of empirical arguments	• Arguments supporting or refuting conjectures about arithmetic relationships that are based on testing specific numerical cases • Explanations that show limitations of empirical arguments (e.g., recognition that not all numbers can be tested)	• Students may believe that conjectures can be justified by example (Knuth, 2002, grades 6-8)	• Justify properties of evens and odds (Bastable & Schifter, 2008, grade 1; Blanton, 2008, grade 5; Carpenter et al., 2003; Kaput, 1998, 1999, grades 1-6)	
3.2) Generalizations about properties of equations and equality are sometimes justified by an algebraic use of numbers, where arguments use specific cases in a way that does not depend on specific cases or numbers.	• Understand how to use numbers algebraically to justify conjectures of the Fundamental Properties • Understand how to use numbers algebraically to justify conjectures about arithmetic generalizations other than the Fundamental Properties	• Arguments supporting conjectures about arithmetic relationships that use numbers algebraically (i.e., as quasi-variables)		• Use specific numbers in general arguments supporting fundmental properties (Carpenter et al., 2003; Russell, Schifter, & Bastable, 2011)	
3.3) Generalizations	• Understand how to use	• Written or oral		• Justify	

[1] The idea of justification (rather than "proof") is used here intentionally. Essentially, these refer to mathematical arguments for why a particular conjectured relationship is a reasonable conjecture.

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about properties of equations and equality are sometimes justified by reasoning with previously established generalizations.	established generalizations to justify conjectures about arithmetic relationships	arguments supporting conjectures about arithmetic relationships that invoke previously established generalizations		generalization by reasoning with previously-established ones (Carpenter et al., 2003)	
3.4) Generalizations about properties of equations and equality are sometimes justified by using representation-based reasoning.	• Understand how to construct and use representations to justify conjectures about arithmetic relationships	• Justifications of conjectures about arithmetic relationships using representation-based reasoning		• Justify using pictorial representations (Bastable & Schifter, 2008; Schifter et al., 2008; Carpenter et al., 2003)	
3.5) Generalizations about properties of equations and equality are sometimes justified by using algebraic arguments based on transforming algebraic expressions.	• Understand how to construct algebraic arguments as appropriate to justify arithmetic generalizations	• Algebraic argument that uses a chain of logic to establish an arithmetic generalization • Justify steps in solving an algebraic equation (e.g., explain the decision to add 5 to both sides)			
3.6) Justifications based on general arguments show that a conjecture is true for all possible cases (e.g., numbers in a specified number	• Understand why general arguments show that a conjecture is true for all numbers in a specified number domain	• Critiques of justifications based on non-general arguments that indicate limitations of the justification • Justifications based on	• Students have difficulty seeing how algebra can be used to justify a general statement about numbers (Kieran, 2007)	• Make general arguments to justify conjectures about arithmetic properties (Carpenter et al.,	

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domain) and thus represent the strongest possible argument.		general arguments and descriptions of affordances of general argument		2003) • Justify conjectures about even and odd numbers in words (Blanton, 2008; Carpenter et al., 2003) • Justify use of the associative and distributive properties to simplify calculations using words and/or manipulatives (Carpenter et al., 2003)	
3.7) While general arguments are needed to prove a conjecture is true, only one counterexample is needed to prove a conjecture false.	• Understand why while general arguments are needed to prove conjectures true, but only one counterexample is needed to prove a conjecture false	• Evaluate appropriateness of using examples to prove true/false conjectures • Use counterexamples to prove a conjecture is false (e.g., $a - b = b - a$)			

STRAND 4. Reasoning with Generalizations

Reasoning with generalizations includes both actions on and actions between representations. This strand describes these actions that might be based on manipulation of symbolic forms using syntactical rules or reasoning with other representations such as graphs or tables to interpret solutions to equations.

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4.1) Equations, inequalities, and expressions often reflect an implicit structure that can lead to insightful solutions to equations (e.g., the equation 3(x + 5) = 12 has a structure that suggests dividing first by 3 rather than multiplying 3 times x + 5 is a more efficient strategy)	• Understand how to think relationally about expressions as objects (e.g., for transforming expressions) • Understand how to think relationally to find solutions to equations and inequalities	• Statements about expressions and equations and observed relationships • Reasoning across the equal sign and use of relational thinking to identify compensation strategies to solve simple equations or "true/false" sentences • Reasoning across equations, including recognition of how one solution relates to another (e.g., x + 7 = 10 and x + 7 = 11)	• Students tend to compute when given the opportunity, as opposed to considering relationships (Schliemann et al., 1998, grade 3)		• Relational thinking open number sentences (Carpenter et al., 2003; Jacobs et al., 2007,;grades 1-6) • Collect 50 cents (Fosnot & Jacob, 2009, grade 2) • T/F sentences (Carpenter et al., 2003, grades 1-6; Fosnot & Jacob, 2009, grades 1-6) • Complete tables for y = f(x) and y = f(x + 1) (Knuth, personal

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					communication)
4.2) Expressions can be rewritten in equivalent forms using Fundamental Properties	• Understand how to simplify algebraic expressions • Understand how to recognize equivalent expressions	• Simplification of algebraic expressions that follow arithmetic properties • Explanations for why expressions are or are not equivalent that rely on arithmetic principles or iconic arguments	• Students tend to make "procedural" errors when simplifying both arithmetic and algebraic expressions: detaching terms from indicated operations, jumping off with the posterior operation, or neglecting order of operations (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996; Linchevski & Livneh, 1999, grades 6-7) • Students have difficulty accepting "lack of closure" in simplifying expressions or giving "answers" (Booth, 1988, grades 8-10) • Students sometimes try to "solve" algebraic expressions • Students make same mistakes simplifying arithmetic expressions		

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			as algebraic expressions (Linchevski & Livneh, 1999, grades 6-7)		
4.3) Expressions can be evaluated to obtain a corresponding numerical value by replacing the variable or variables with the indicated quantity or quantities	• Understand how to substitute numerical values into variables in an expression and compute the numerical result	• Substitution of variables with specific numbers • Accurate computations of expressions where variables have been replaced with specific numbers			
4.4) Equations can be rewritten in equivalent forms using Fundamental Properties and Properties of Equations to yield mathematical insights and solutions.	• Understand how to recognize equivalent equations • Understand the notion of "undoing" in solving equations and the relationships between inverse operations • Understand how to solve simple (one-step) equations and multi-step equations involving a single or repeated variable	• Application of properties to rewrite equations in equivalent forms • Use of properties to argue that equivalence relationships have or have not been preserved • Statements that indicate understanding "doing the same thing to both sides" preserves the equivalence, even if the equation is complicated rather than simplified by this action • Application of various strategies to solve open number sentences (e.g.,	• Students have difficulty recognizing that performing the same operation on both sides of an equation preserves the equivalence relationship (Alibali et al., 2007; Knuth et al., 2005; Knuth et al., 2008; Steinberg et al., 1990) and are less likely to accept adding to both sides than subtracting from both sides (O'Rode, 2003), grades 6-8 • Students tend to compute when given	• Make judgments about equation equivalence (Knuth et al., 2005; Knuth et al., 2008; O'Rode, 2003; Schliemann et al., 1998; Steinberg et al., 1990), grades 6-8 • Understand the notion of "undoing" in solving equations and the relationships between +/- and x/÷ • Solve simple equations	• Verbal equivalent equations (Schliemann et al., 1998, grade 3) • Judging equation equivalence (Knuth et al., 2005; Knuth et al., 2008; Steinberg et al., 1990, grades 6-8) • Three choices, Messing up equations

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		compute each side and count up/down, relational thinking, guess and test, unwinding) • Reflection on different symbolic strategies and their efficiencies and connections (Rittle-Johnson & Star, 2007)	the opportunity, as opposed to considering relationships (Schliemann et al., 1998, grade 3) • Students have difficulty operating on unknowns in solving linear equations (Herscovics & Linchevski, 1994, grade 7) • Students have difficulty solving linear equations with negative integers (Vlassis, 2002, grade 8)	(Carpenter et al., 2003) • Solve equations with repeated variables (Carpenter et al., 2003) • Solve equations with multiple variables (Carpenter et al., 2003) • Solve equations of the form ax + b = c (Brenner et al., 1997; Johanning, 2004; Knuth et al., 2006; Stacey & MacGregor, 2000; Swafford & Langrall, 2000) • Solve equations of the form ax + b = cx + d (Kieran & Sfard, 1999; Stacey & MacGregor, 2000; Yerushalmy, 2000)	(O'Rode, 2003, grade 8) • Open number sentences (Carpenter et al., 2003) • Number sentences with repeated variables (Carpenter et al., 2003, grades 3-6) • Number sentences with multiple variables (Carpenter et al., 2003, grades 3-6) • Mice in cages (Carpenter et al., 2003, grades 3-6) • Equation writing and solving (Swafford & Langrall, 2000, grade 6)

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					• Pet, Candy bar (Johanning, 2004, grades 6-8) • Stone piles (NRC, 2001, p. 257) • Triangle, Mark, Bus (Stacey & MacGregor, 2000, grades 9-10) • Equation writing and solving (Brenner et al., 1997, grades 7-8) • Equation solving (Knuth et al., 2006, grades 6-8) • Cheapest club, Parking lots, Bonus (Yerushalmy, 2000, grades 7-9) • Number (Stacey & MacGregor,

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					2000, grades 7-10)
4.5) The solutions of an equation or inequality are the values of the variables that make the equation or inequality true.	• Understand that solutions of an equation or inequality are the values of the variables that make the equation or inequality true • Understand how to substitute a value into an equation or inequality to determine if it is a solution • Understand how to transform an equation or inequality into equivalent forms to isolate its solution	• Statements indicating that values that make an equation true are solutions of the equation • Substitution of values into variables to check solutions to equations	• Students may view the "solution" as the number to the right of the equal sign (Carpenter et al., 2003, grades 3-6)		
4.6) Graphical and tabular representations of equations can be used to interpret solutions to equations and systems of equations	• Understand how to solve linear equations and systems of linear equations using graphical solution strategies • Understand how to solve linear equations and systems of linear equations using tables	• Graphs of the lines y = ax + b and y = cx + d • Recognition that the x-value at the intersection gives the solution to the system of equations • Recognition of what intersecting, parallel, and coincident lines mean in terms of solutions • Articulation of the	• Students have difficulty seeing connections between symbolic and graphical solution strategies (Knuth, 2000, grades 9+) • Students may not understand that graphs of lines extend infinitely beyond the printed page	• Solve equations of the form ax + b = cx + d using graphical solution strategies (Kieran & Sfard, 1999, grade 7)	• See Kieran & Sfard (1999) for tasks (grade 7) • Best Deal (Brizuela, 2003, grades 2-3) • Equation-graph tasks (Knuth, 2000, grades 9+)

CONSTRUCT (This column contains constructs within the Reasoning strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
		connection between symbolic and graphical representations and solution strategies			
4.7) Graphical representations of inequalities can be used to interpret solutions to inequalities.	• Understand how to solve linear inequalities using graphical solution strategies	• Recognize which region of the graph corresponds to the specific inequality and that that region represents the solution to the inequality • Recognize that intersecting regions of a graph corresponds to a solution to a system of linear inequalities			

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