PROGRESSION FOR DEVELOPING ALGEBRA UNDERSTANDING THROUGH FUNCTIONAL THINKING ACROSS GRADES 3-7:

This curricular progression is intended to develop algebra understanding through a study of functions. Functional thinking involves building, representing, and reasoning with and about functions. It can serve as an important point of entry into algebra because it involves generalizing relationships between co-varying quantities; representing those relationships, or functions, in multiple ways using natural language, formal algebraic notation, tables and graphs; and reasoning fluently with these representations in order to interpret and predict function behavior.

SUMMARY OF DEVELOPMENT OF IDEAS ACROSS GRADES 3-7:

The following summarizes a proposed curricular progression for the development of algebra understanding through functional thinking in grades 3-7. The progression is based on what we know from research that children can reasonably accomplish. It is not intended to be unique, but represents one progression for developing algebra understanding through the study of functions. It focuses on functions that involve covariation between quantities (not functions in a general sense). It uses tasks that are situated in appropriate problem contexts to motivate the development of functions. We consider only functions with two varying quantities and we use functional situations with idealized patterns (e.g., "perfectly" linear, quadratic, or exponential). We have opted not to include a focus on messy data because our view is that it is important for students to develop a strong understanding of various function types prior to engaging in approximations of these types. While our problem contexts are idealized, they may still be based on real-world scenarios. Moreover our use of symbols is restricted to the use of literal symbols (not non-literal symbols such as Δ, since some research shows these can be problematic and can lead to misconceptions).

Third Grade:

In 3rd grade, students focus on linear functions to begin their development of an understanding of different types of relationships (recursive, covariational, and correspondence), how to represent these patterns and relationships in multiple ways (words, variables, tables and graphs), and how to justify generalized relationships (e.g., reasoning from the function table or problem context). The emphasis is on correspondence relationships, which can be expressed in words or symbols as function rules. Linear functions focus on one operation and build from multiplicative (i.e., $y = mx$) to additive (i.e., $y = x + b$). Students are also introduced to basic concepts associated with constructing coordinate graphs to represent a linear functional relationship. Attention is given to issues such as scale, predicting placements for unknown points (e.g., predicting where to place the "next" point), qualitative behavior of the graph (e.g., increasing or decreasing, idea of "steady") and representing discrete data. Students begin to informally connect functions and equations by examining functions for which the dependent variable is a specific value. Through this, they explore reversibility as a precursor to inverse operations by actions of 'doing and un-doing' on function rules. Multiplicative functions of the form $y = mx$ are used to explore proportional reasoning.

Types of Functions: $y = ax$ and $y = x + b$ (for positive integers a and b)

Core Actions:

- generate data and organize in a function table;
- identify variables (including as number of object/magnitude of quantity, not object/quantity) and their role as varying quantity;
- identify a recursive pattern and describe in words; use pattern to predict near data;
- identify a covariational relationship and describe in words;
- identify a function rule and describe in words and variables;
- use function rule to predict far function values;
- examine meaning of different variables in a function;

- develop a justification for why the function rule works using the function table (that is, substituting values from the function table in the rule) and by reasoning from the problem context;
- recognize that corresponding values in a function table must satisfy the function rule. That is, when function variables are substituted with corresponding values from the table, the result must be a true equation;
- construct a coordinate graph of a functional relationship and attend to how discrete data are represented and issues of scale (e.g., equal spacing between units, extending the graph beyond the window);
- use multiplicative relationships to reason proportionally about data (e.g., If 2 pieces of candy cost 10 cent, how much would 4 pieces cost?);
- reversibility: use either the function table or function rule to determine the value of the independent variable given the value of the dependent variable.

Fourth Grade:

In 4th grade, students strengthen their understanding of different types of relationships of linear functions of the form $y = mx + b$, representations of relationships in words, variables, tables, and graphs, and justifications of generalized relationships. They extend their work with functions to include quadratic relationships, focusing initially on relationships of the form $y = x^2$ and extending this more generally to those of the form $y = x^2 + b$ (for positive integers b). Students continue to informally connect functions and equations by examining functions for which the dependent variable is a specific value. Through this, they continue to develop an intuition about reversibility, as a precursor to inverse operations, by actions of 'doing and un-doing' on function rules. They deepen their knowledge of functions by learning to interpret and predict the qualitative behavior of a single function (linear or quadratic) by inspecting its graph and function table; examining qualitative growth differences in functions by looking at their graphs and function table; and interpreting graphs of two linear functions in order to solve mathematical situations (e.g., Best Deal). Concepts are sequenced so that students first informally explore these ideas using more familiar linear functions. The work of predicting and interpreting function behavior, examining growth patterns in tables and graphs, and interpreting the graphs of two functions to solve mathematical situations, is then extended to include simple quadratic relationships.

Types of Functions: $y=ax$, $y=ax + b$, $y = x^2$, $y =x^2+ a$ (for positive integers a, b)

Core Actions:

- generate data and organize in a function table;
- identify variables (including as number of object/magnitude of quantity, not object/quantity) and their role as varying quantity;
- identify a recursive pattern and describe in words; use pattern to predict near data;
- identify a covariational relationship and describe in words;
- identify a function rule and describe in words and variables;
- use function rule to predict far function values;
- examine the meaning of different variables in a function;

- develop a justification for why the function rule works by reasoning from the problem context or the function table;
- recognize that corresponding values in a function table must satisfy the function rule. That is, when function variables are substituted with corresponding values from the table, the result must be a true equation;
- construct a coordinate graph and examine issues of scale and data representation;
- use multiplicative relationship to reason proportionally about data;
- reversibility: use either the function table or function rule to determine the value of the independent variable given the value of the dependent variable;
- predict a far data value by thinking intuitively about how the function is changing (increasing/decreasing; how quickly?) from table and graph; check using the function rule;
- interpret graphs of two linear functions to solve a problem situation;
- describe growth patterns informally for linear and quadratic functions by looking at co-variation (how does the dependent variable change given a unit change in independent variable);
- compare growth patterns for linear and quadratic functions (e.g., which grows faster and why?); use graphs and function tables to explain differences in growth and why a particular function grows faster;
- identify qualitative connections between the growth pattern in the function table and the shape of the graph for linear and quadratic functions (e.g., what does the co-variational relationship observed in the function table mean for the shape of the graph?);
- interpret function behavior for linear and quadratic functions depicted in tables or graphs to solve a problem situation (e.g., which is the better diet and why?);

Fifth Grade:

In 5th grade, students continue to strengthen their understanding of different types of relationships, representations of these relationships, and justifications of generalized relationships. They extend their work with linear and quadratic functions to include exponential relationships. They deepen and extend their knowledge of functions by interpreting and predicting the behavior of a linear, quadratic or exponential function; examining growth differences in different types of functions (linear, quadratic and exponential) by looking at their graphs, function tables, function rules, and connections among these representations; interpreting graphs of two functions of different types (e.g., one linear and one quadratic) in order to solve mathematical situations; and interpreting the behavior of functions represented through qualitative graphs. Ideas are sequenced so that students initially interpret and predict function behavior and growth in the context of more familiar linear and quadratic functions. These ideas are then explored using simple exponential functions of the form $y = a^x$. Comparisons and interpretations of function behavior are made across all three function types, continuing previous work by comparing growth patterns evidenced by graphs and tables, and extending this by predicting shapes of graphs based on growth patterns in function tables. As part of this, students are asked to think about problem situations in which functions might be increasing or decreasing, and to determine this behavior from graphs or tables. Finally, qualitative graphs are introduced to strengthen students' ability to coordinate two variables in interpreting function behavior and to introduce the transition from discrete to continuous variables.

Types of Functions: $y=ax$, $y=ax+b$, $y=x^2$, $y=x^2+b$, $y=a^x$ (for positive integers a, b)

Core Actions:

- generate data and organize in a function table;
- identify variables (including as number of object/magnitude of quantity, not object/quantity) and their role as varying quantity;
- identify a recursive pattern and describe in words; use to predict near data;
- identify a covariational relationship and describe in words;
- identify a function rule and describe in words and variables;

- use the function rule to determine far data points;
- develop a justification for why the function rule works by reasoning from the problem context or the function table;
- recognize that corresponding values in a function table must satisfy the function rule. That is, when function variables are substituted with corresponding values from the table, the result must be a true equation;
- construct a coordinate graph and attend to scale and data representation;
- reversibility: use either the function table or function rule to determine the value of the independent variable given the value of the dependent variable;
- for linear functions, use multiplicative relationship to reason proportionally about data (e.g., If 2 pieces of candy cost 10 cent, how much would 4 pieces cost?);
- predict a far data value by thinking intuitively about how the linear function is changing (increasing/decreasing; how quickly?) from table and graph; check using the function rule;
- describe growth patterns informally by looking at co-variation in linear, quadratic, and exponential functions (How does the dependent variable change given a unit change in independent variable?); use descriptions to predict or describe shape of each graph, including which graph might grow fastest, etc.
- check predictions of shapes of graphs for linear, quadratic and exponential functions by comparing to actual graphs;
- develop qualitative connections between growth pattern in function table and shape of graph for linear, quadratic, and exponential functions (e.g., what does covariational relationship mean for shape of graph?);
- predict far data values by thinking informally about how linear, quadratic, and exponential functions are changing (increasing/decreasing; how quickly?) from table and graph; check using the function rules;
- compare growth patterns for linear, quadratic, and/or exponential functions (e.g., which grows faster and why?); use graphs, function tables, and function rules to describe differences in growth and why a particular function grows faster;
- interpret function behavior for linear, quadratic, and exponential functions depicted in tables or graphs to solve a problem situation (e.g., which is better diet and why?);

- examine the nature of continuous variables and how data are represented in qualitative graphs of piecewise functions;
- interpret the behavior of a piecewise function represented by a qualitative graph at various regions of the graph (e.g., what's happening on horizontal segments, what's happening on segments that are not horizontal, what's happening at corners);
- construct a narrative (story) to match a qualitative piecewise graph
- given a story (narrative) of linear and/or nonlinear motion, construct a qualitative graph to match the motion depicted in the story; identify variables and describe how parts of the graph are reflected in the narrative and why the shape of the graph accounts for the narrative.

Sixth Grade*:*

In 6th grade, students strengthen their knowledge of linear and nonlinear functions through a focus on representing functions in multiple ways, comparing different representations, using different representations to interpret, predict and compare behavior in functions, and using representations (particularly, for linear relationships) to solve mathematical situations. They extend their understanding of linear functions by examining characteristics of linear data and by identifying informally particular features of linear functions (i.e., slope, y-intercept) and how these are reflected in the function's rule, graph and table. They are introduced to the idea of slope as a unit rate and to proportional relationships as a special case of linear functions.

Seventh Grade:

In 7th grade, students focus primarily on linear relationships, strengthening their understanding of how to represent linear functions in multiple ways; comparing and translating among different representations to interpret and predict function behavior; understanding what constitutes linearity; and using various representations to solve problem situations. They extend their understanding of connections between proportional relationships and linear functions of the form $y = mx$ to include differences between proportional relationships and other relationships (e.g., inverse proportionality). They continue to develop their understanding of connections between slope and y-intercept of linear functions as represented in multiple representations (i.e., tables, graphs, and symbolic rules). They also begin to apply covariational reasoning to solve linear problem situations. Finally, they are introduced to the concept of a family of (linear) functions and explore the role of variable as parameter to represent a family of functions.