FUNCTIONAL THINKING

Functional thinking is the process of building, describing, and reasoning with and about functions. It can serve as an important point of entry into algebra because it involves generalizing relationships between quantities; representing those relationships, or functions, in multiple ways using natural language, formal algebraic notation, tables and graphs; and reasoning fluently with these representations in order to interpret and predict function behavior.

STRAND 1. Relationships (Generalizing Relationships)

Relationships between two co-varying quantities may be linear or nonlinear and may be expressed using recursive patterns[1], covariational relationships, or correspondence rules. This strand identifies content related to the characteristics of types of relationships that might be generalized in functional situations.

CONSTRUCT (This column contains constructs within the Relationships strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** **(This column contains research-based tasks.)**
1.1) Recursive patterns describe variation in a single sequence of values. A recursive pattern indicates how to obtain a number in a sequence given the previous number or numbers.	• Understand how to identify and describe recursive patterns in words and use these patterns to predict near data • Understand how to identify a pattern appropriately as recursive	• Descriptions of recursive patterns in words • Predictions of "next" function value using recursive pattern • Characterization of recursive pattern as recursive		• Describe recursive patterns in words (Martinez & Brizuela, 2006; Blanton, 2008, grades 1-6) • Predict "next" function value using recursive pattern Blanton, 2008,	The following tasks apply to many of the constructs in this strand and others: LINEAR • Dogs and eyes (Blanton & Kaput, 2005, p. 38, grades K-6)

[1] Include tasks that help students see limitations of recursive patterns (e.g., use of large numbers). This should be natural conversation in classroom, but we are not asking students in elementary grades to specifically know the vocabulary of different types (e.g., describe a pattern as recursive)

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				grades 2-5) • Generate recursive and explicit function rules in words given tasks that ask for near and far predictions and about general iconic or numeric relationships – students realized far predictions were difficult when reasoning recursively. Some reasoned from pictures (e.g., rows of theater seats), others looked at number patterns (Lannin et al., 2006, grade 6)	• Piggy banks (Brizuela & Lara-Roth, 2002, p. 312, grade 2) • Best deal (Brizuela & Earnest, 2008, p. 281, grades 2-4) • Trapezoid tables (Blanton & Kaput, 2003, p. 75; Blanton, 2008, p. 36) • T-shirts (Blanton & Kaput, 2003, p. 72) • Candy boxes (Carraher et al., 2008, p. 238, grades 2-4) • Visual growth patterns—continue and generalize (Warren & Cooper, 2008, p. 176, grade 3) • Dinner tables (Martinez & Brizuela, 2006, p. 288, grade 3) • Tower (NCTM, 2000, p. 160) • Saving for a bike
1.2) Covariational relationships describe how two quantities vary in relation to each other as an expression of the change in one quantity given a (unit) change in the related quantity (Confrey & Smith, 1991).	• Understand how to identify and describe covariational relationships in words (e.g., as x increases by 1, y increases by…) • Recognize a relationship appropriately as covariational	• Description of covariational relationships in words (e.g., "When x increases by 1, y increases by 3") • Characterization of covariational relationship as covariational		• Describe covariational relationships in words (e.g., "When x increases by 1, y increases by 3") (Blanton, 2008)	

1.3) A correspondence relationship is a correlation between corresponding pairs of variables, designated as either the independent variable or dependent variable, typically expressed as a function rule[2] (Smith, 2008).	• Understand how to identify and describe correspondence relationships, or function rules, using words or variables. • Recognize a relationship appropriately as a correspondence • Be able to distinguish the independent variable and dependent variable in a function rule	• Description of function rule in words or variables • Characterization of correspondence relationship as function rule • Characterization of variables as, appropriately, independent or dependent	• Students may use "whole object reasoning" when finding function rules (using proportional reasoning to come up with y = mx when function is y = mx + b) (Moss et al., 2008, grade 4)	• Express function rule in words and variables (Blanton, 2008, grades 3-5) • Express linear functions using tables and graphs (Blanton, 2008; Brizuela & Earnest, 2008, grades 3-6) • Identify correspondence rules of the form y = mx + b from visual patterns (e.g., trapezoid table problem) (Moss et al., 2008, grade 4) • Express correspondence rules in linear functions in words (Martinez & Brizuela, 2006; Blanton 2008, grades 2-5) • Express correspondence rules in pictorial linear relationships in words (Warren & Cooper, 2008, grade 3) and symbols (Blanton, 2008,	(Blanton, 2008, p. 72) • Two NAEP items giving students table of values and asking them to find the rule (Kenney & Silver, 1997, p. 271, grade 4) • NAEP item asking students how many tacks needed to hang given number of pictures (Kenney & Silver, 1997, p. 272, grade 4) • Findings rules, inputs, and outputs from linear function tables (Warren et al., 2006, grade 3) QUADRATIC: • Growing snake (Blanton & Kaput, 2005, p. 37; Blanton, 2008, p. 42,

[2] We use the term function rule here to refer specifically to a correspondence relationship. Note that in this progression we are only considering functions whose rules can be expressed in closed form and where the relationship is between two co-varying quantities.

				grade 3) • Express correspondence rules, inputs, and outputs in linear functions using arrows (Warren et al., 2006, grade 4) • Express correspondence rules in exponential relationships in tables, words, and symbols (Blanton, 2008, grade 5) • Express explicit function rule in words given tasks that ask for near and far predictions and about general iconic or numeric relationships (Lannin et al., 2006, grades 6-9) • Express explicit rules for geometric patterns in words (Radford, 2000, grade 8)	grade 3) • Handshake (Blanton, 2008, p. 59, grades 1-6) • Triangle dot (Blanton, 2008, p. 64)

1.4) Linear relationships in co-varying data are relationships characterized by a constant rate of change, that is, a unit change in the independent variable results in a constant change in the dependent variable.	• Understand that linear relationships are characterized by a constant rate of change, that is, a unit change in the independent variable results in a constant change in the dependent variable (or, more generally, if the independent variable changes by *a*, then the dependent variable changes by *ma*) • Understand differences in exponential, quadratic and linear relationships based on co-variation (change) in data (CMP grade 8-Growing, Growing, Growing)	• Characterizations of relationships depicted in problem situations, tables, graphs, or function rules for which co-varying data has a constant rate of change, as linear (CMP grade 7-Moving Straight Ahead) • Identify relationships appropriately as linear, quadratic, or exponential based on co-variation	• Students tend toward linearity when reasoning about relationships • Students sometimes have difficulty accepting constant functions (for many x, the same y)		
1.5) Linear relationships are considered to be proportional relationships if the ratio of the dependent variable to the independent variable is constant.	• Understand a proportional relationship as a special case of linear relationships • Understand what qualifies a linear relationship as proportional • Understand that for proportional	• Describe proportional relationships as linear • Describe linear relationships appropriately as proportional • Identify the constant of proportionality, *k*, as the ratio of the independent variable to the dependent			

	relationships, *k* is called the *constant of proportionality* and represents both the ratio of the dependent variable to the independent variable and the slope of the line y=kx • Distinguish proportional relationship ($y = mx$) from inverse proportionality ($xy = k$, or $y = k/x$)	variable • Identify the constant of proportionality, *k*, as the slope of the line y=kx			
1.6) Quadratic relationships in co-varying data are nonlinear relationships characterized by rates of change that change at a constant rate.	• Understand that quadratic relationships are nonlinear and are characterized by rates of change that change at a constant rate • Understand differences among exponential, quadratic and linear relationships based on co-variation in data (CMP grade 8-Growing, Growing, Growing)	• Identify the patterns of change for quadratic relationships (CMP grade 8-Frogs, Fleas, and Painted Cubes) • Identify quadratic relationships as nonlinear • Characterize quadratic change as change at a constant rate • Identify differences among exponential, quadratic and linear relationships based on co-variation in data			

1.7) Exponential relationships in co-varying data are nonlinear relationships characterized by a rate of change that increases (exponential growth) or decreases (exponential decay) over the domain.	• Understand that exponential relationships are nonlinear and are characterized by an increasing or decreasing rate of change (rate of change getting "bigger and bigger" or "smaller and smaller") • Understand differences among exponential, quadratic and linear relationships based on co-variation in data (CMP grade 8-Growing, Growing, Growing)	• Identify the patterns of change for exponential relationships • Identify exponential relationships as nonlinear • Characterize exponential change as relationships with an increasing or decreasing rate of change • Identify differences among exponential, quadratic and linear relationships based on co-variation in data			
1.8) A family of (linear) functions is a set of functions with similar patterns of change.	• Understand how to represent a family of (linear) functions using parameters • Understand different roles of variable (variable as varying quantity, variable as parameter) in representation of family of functions • Understand that patterns of change determine whether functions belong to the same or different family	• Characterize a family of (linear) functions in generalized form using parameters • Identify the different roles of variables in a family of functions as, appropriately, varying quantity or parameter • Identify functions as belonging to the same or different family of functions			

STRAND 2. Representations (Representing Generalizations)

Functional situations can be represented in a variety of forms, including words, symbols, tables and graphs. This strand addresses types of representations that are used to model, describe or reason with co-varying quantities and particular forms of these representations that reflect linear or nonlinear relationships.

CONSTRUCT (This column contains constructs within the Representations strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
2.1) Co-varying relationships can be depicted through problem situations. Co-varying data can then be generated from these situations and kept track of through a variety of representations, from iconic to symbolic.	• Understand how to model the mathematical situations in a function task, use appropriate forms to represent data (e.g., iconic, symbolic) and organize data in a way that facilitates functional thinking	• Written inscriptions (e.g., diagrams) that depict specific values for varying quantities being generated from the problem situation, the specific values for these varying quantities, and an appropriate representation of connections between corresponding values	• Students initially have difficulty understanding the need to keep the independent variable explicit in co-varying data (Blanton, 2008, grades 1-5) • Students initially have difficulty understanding efficient or strategic ways to keep track of data for a particular varying quantity and that or how a specific value of one varying quantity relates to another (Blanton, 2008, grades 1-5) • Student have more	• Construct table of values that appropriately represents a particular story (e.g., time vs distance) (Tierney & Monk, 2008, grades 3-5)	The following tasks apply to many of the constructs in this strand: • Best Deal (Brizuela & Ernest, 2008, grades 2-4) • Tasks relating equations and graphs (Knuth, 2000, grades 9+) • Generate table and tell a story it represents (Tierney & Monk, 2008, p. 196, grades 3-5) • Walk-a-thon context

			difficulty representing nonlinear functions using symbols and graphs than they do linear functions (Kieran, 2007) • Students are generally more successful in using one representation than translating among representations; they prefer tables of values first before developing facility with graphs and equations (Kieran, 2007)		(Kalchman & Case, grade 6) • Situations presented in words, students asked to make near and far predictions, generalize in words and variables, and use equation to answer a question (Swafford & Langrall, 2000, grade 6) • Translation among reps (see Kieran & Sfard, 1999, pp. 6-10, grade 7) • Brenner et al. (1997) assessment items: function word problem test, word problem representation test, word problem solving test, equation solving test (p. 674, grades 7-8)
2.2) Function tables can be used to represent corresponding values of the independent and dependent variables.	• Understand how to construct a function table, including its constituent parts and their placement	• Constructions of function tables with appropriate headings (in words or variables) and correct placement of data.	• Students have difficulty accounting for two variables simultaneously. • Students tend to think about pictorial linear patterns recursively rather than keeping the figure number, or value of the independent variable, explicit (Warren & Cooper, 2008, grade 3)	• Use function tables to organize co-varying data (Blanton, 2008, grades 1-5) • Represent additive relationships in function tables (Brizuela & Lara-Roth, 2002, grade 2)	

2.3) Coordinate graphs can be used to visually depict a functional relationship in co-varying quantities appropriate to scale.	• Understand how to construct coordinate graphs to represent functions, including appropriately represent discrete or continuous data to scale and construct axes and label with variables appropriately • Understand that the function's graph extends beyond the region displayed • Understand how a graph represented continuously differs from a graph of discrete data	• Constructions of coordinate graphs with appropriate representations of data and scale, and appropriate axes and labels • Description of function behavior based on parts of graph not displayed • Explanation for use of points to represent discrete data or lines/curves to represent continuous data	• Students have misconceptions relating to the form of the graph. For example, they will connect discrete data with a continuous line. They also have misconceptions about scale and understanding the need to construct equal units on the coordinate axes.	• Construct coordinate graphs to represent co-varying data (Blanton, 2008, grades 3-5) • Construct graph based on plotting points from a table of values (Kieran & Sfard, 1999, grade 7)	• The table on the left represents specific values of the function f(x) = x^3 - 3x^2 + 2x. Fill in the table on the right, which represents the function y = x^3 - 3x^2 + 2x + 1 (Moschkovich et al., 1993, p. 81). • What are possible values for the slope of a line that lies entirely in the shaded region? (Moschkovich et al., 1993, p. 82) • Why is the graph of y = 3x steeper than the graph of y = 2x? What about y = 4x, y = 5x, y = 10x? (Moschkovich et al., 1993, p. 83) • Match linear equations to graphs (Leinhardt et al., 1990, pp. 17-18) • CMP Tasks from Moving Straight
2.4) Function rules can be described in words using natural language or through the use of variables as varying quantities.	• Understand how to express a linear, quadratic, or exponential relationship as a function rule between two varying quantities in natural language or variables • Understand that linear functions can be represented symbolically in the form y = mx + b (or equivalent) • Understand that linear functions that are	• Descriptions of function rules that use natural language and that account for both variables • Symbolic descriptions of function rules that use variables to represent varying quantities and that account for both variables	• Students have an easier time stating generalizations about functions orally than in writing (Warren & Cooper, 2008, grade 3) • Students sometimes have difficulty using precise language to describe a pattern (Warren & Cooper, 2008, grade 4) • Students will express		

	proportional relationships can be represented symbolically in the special form $y = kx$ (where $b = 0$) • Understand that quadratic functions can be represented symbolically in the form $y = ax^2 + bx + c$ (or equivalent) • Understand that simple exponential functions can be represented symbolically in the form $y=a^x$ (or equivalent) *(The above does not necessarily mean students have to know the general form per se, but be able to express functions in appropriate forms)* • Understand how to use the function rule to determine values of one variable given values of the corresponding variable (e.g., reversibility, substitution) • Understand that a function rule is linear, quadratic or exponential based on its symbolic form	• Linear functions expressed in the form $y=mx +b$ (or equivalent) • Quadratic functions expressed in the form $y = ax^2 + bx + c$ (or equivalent) • Exponential functions expressed in the form $y=a^x$ (or equivalent) • Equations created by substituting, in the function rule, a specific value for one of the variables and that have been solved for the unknown variable • Identify a function rule as linear, quadratic, or exponential based on its symbolic form	function rules using a mixture of natural language and symbols (e.g., "the number of people is 3x + 2") or just as an expression (e.g., "3x + 2") where the connection between variables is not made explicit (i.e., 'y = 3x + 2') (Blanton, 2008, grades 3-5)		Ahead (grade 7 - linear) and Frogs, Fleas and Painted Cubes (grade 8 - quadratic)

2.5) Qualitative graphs allow for functions to be represented on coordinate axes that do not contain units so that function behavior can be analyzed holistically.	• Understand that graphs can be represented qualitatively, without quantifying marks or information	• Recognize that graphs represented without quantifying information are valid representations of co-varying data			• Height of plants vs time qualitative graph interpretation (Kaput, 1998, p. 14; Tierney & Monk, 2008, p. 193) • Size of pots vs height of plants qualitative graph interpretation (Leinhardt et al., 1990, p. 11) • Distance vs speed of racing car qualitative and quantitative interpretations (Leinhardt et al., 1990, p. 21) • Interpret time vs number of students in building during fire drill graph (Leinhardt et al., 1990, p. 29) • Distance vs time qualitative

					graphs—which represent journeys? (Leinhardt et al., 1990, p. 39) • Height of plants vs time qualitative graph interpretation (Kaput, 1998, p. 14; Tierney & Monk, 2008, p. 193)

STRAND 3. Justifications (Justifying Generalizations)

Generalized relationships in co-varying data can be justified[3] using different strategies. This strand describes how justification occurs in the context of FT.

CONSTRUCT (This column contains constructs within the Justification strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
3.1) A function rule is sometimes justified by using substitution to test specific pairs of corresponding quantities in the function rule.	• Understand how to substitute specific pairs of corresponding quantities in a function rule and check validity	• Explanations of how data in a function table supports a conjectured function rule	• Students are more apt to have success generalizing and justifying when tasks are tied to visual patterns rather than just numeric information, in part due to lack of operation sense (Lannin, 2005, grade 6)		
3.2) A function rule is sometimes justified by reasoning from the context of the problem situation.	• Understand how to use the problem context to justify a function rule.	• Explanations of how the problem context supports a conjectured function rule			

[3] The idea of justification (rather than 'proof') is used here intentionally. Essentially, these refer to mathematical arguments for why a particular conjectured relationship (in this case, a function rule) is a reasonable conjecture.

3.3) A function rule can sometimes justified by connecting salient features of other representations (e.g., graph, function table) to the function rule.	• Understand how to relate features of a graph or table to the function rule to justify the rule (e.g., relate slope and y-intercept of graph of linear function to function rule; look at table to identify slope and y-intercept)	• Explanations of how features of a graph or table are connected to the function rule			

STRAND 4. Reasoning with Generalizations

Reasoning with generalizations includes both actions on and actions between representations. These actions might be based on manipulation of symbolic forms using syntactical rules or reasoning with other representations such as graphs or tables to understand function behavior.

CONSTRUCT (This column contains constructs within the Reasoning with Generalizations strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
4.1) Representations such as graphs (quantitative or qualitative), tables, and function rules can be used to interpret and predict behavior of a function or set of functions	• Understand how to describe growth patterns informally by looking at co-variation (how does dependent variable change given a unit change in independent variable) • Understand how to use the shape of graph to interpret function behavior, predict far data values (e.g., if it's linear shape, this says something about how function will grow/decay - increasing/decreasing? How quickly?), recognize linear, quadratic or exponential functions (see, e.g.,	• Qualitative descriptions of growth patterns in linear, quadratic and exponential functions based on function tables • Predictions of function behavior at far data points based on shape of graph; (Connected Mathematics Variables and Patterns- Grade 7) • Characterizations of functions as linear, quadratic or exponential based on shape of graph and descriptions of	• Students may only accept graphs that follow obvious patterns as functions. For example, students may not see piecewise functions, discontinuous functions, or otherwise unfamiliar functions as functions (Leinhardt et al., 1990) • Students have difficulty coordinating two variables to interpret the shape of a graph for distance/time graphs or rate/time graphs. They initially focus on coordinate points to think about behavior, rather than	• Work with data in a table without accompanying pictures of the situation (Blanton & Kaput, 2005, grades 3-5) • Compare linear functions using tables and graphs (Brizuela & Earnest, 2008, grades 2-4) • Construct a table of values that appropriately represents a story (e.g., time vs distance) (Tierney & Monk, 2008, grades 3-5)	

	CMP, grade 8-Frogs, Fleas, and Painted Cubes), and compare functions to explain differences in growth (e.g., which grows faster and why) • Understand how to use growth patterns (covariation) in tables to interpret function behavior and predict far data values (e.g., by looking at covariation, think about how function is behaving at near and far data points – increasing/decreasing? How quickly?), predict shape of graph, recognize linear, quadratic or exponential functions (see, e.g., CMP, grade 8-Frogs, Fleas, and Painted Cubes), and compare functions to explain differences in growth (e.g., which grows faster and why) • Understand how to use function rule to interpret function behavior and predict far data values (e.g., understanding how	features of graph that indicate this • Comparative descriptions of growth in two or more functions and justifications for descriptions based on information in graphs • Predictions of function behavior at far data points based on information in function tables • Characterizations of functions as linear, quadratic or exponential based on information in function tables and descriptions of features of tables that indicate this • Comparative descriptions of growth in two or more functions and justifications for descriptions based on information in function tables • Predictions of function behavior at far data points based on function rule	looking at the shape of the graph (Burke, 2010, grade 4) • Students' tend to interpret graphs iconically (Leinhardt et al., 1990) • Students tend to interpret graphs in a point-wise manner rather than holistically. They can reason about the meaning of particular points but have trouble viewing the big picture • Students tend to interpret graphs iconically; They view graphs as pictures rather than thinking about the meaning of x and y and their relationship (e.g., in graphs of distance vs time, they may view increase as "going up hill")	• Compare two linear functions using tables or graphs (Anderson, 2008; Yerushalmy, 2000, grades 7-9) • Construct table of values and a graph for any function of the form y = f(x) for x = 0 or any positive integer (Kalchman & Case, grade 6) • Grade 6: Oral or written descriptions of second-order features of tables and graphs (e.g., common differences) (Kalchman & Case, grade 6) • Describe of important features of functions from tables, graphs, equations, and verbal rules such as linearity, steepness, and y-intercept	

	functions grow with linear, quadratic or exponential function rules), recognize linear, quadratic or exponential functions (see, e.g., CMP, grade 8-Frogs, Fleas, and Painted Cubes) and compare functions to explain differences in growth (e.g., which grows faster and why) • Understand how to interpret points or regions of interest in a graph (e.g., what does it mean, mathematically and in terms of the problem context, for graphs to intersect; whether system of linear functions have graphs of lines that intersect, are parallel, or are the same line) • Understand how to characterize growth in terms of steepness of lines for linear functions • Understand how to interpret function tables, graphs or function rules to solve problem	• Characterizations of functions as linear, quadratic or exponential based on function rule and descriptions of features of graph that indicate this • Comparative descriptions of growth in two or more functions and justifications for descriptions based on function rule • Descriptions of salient points or regions of interests in a graph and their interpretation, either mathematically or based on the problem context • Descriptions of growth of linear functions in terms of steepness of lines • Interpretations of information in graphs, tables, or function rules that solve a problem situation • Characterizations of linearity in problem		(Kalchman & Case, grade 6) • Compare of two linear functions using tables or graphs (Yerushalmy, 2000, grades 7-9) • Interpret the meaning of qualitative linear graphs (e.g., comparisons of two lines with different slopes and y-intercepts and interpret these differences in context) (Kaput, 1998; Tierney & Monk, 2008, grades 3-5)	

	situations (CMP, grade 7-Moving Straight Ahead) • Understand how to recognize linearity in problem situations • Understand how to interpret a story scenario to construct a qualitative graph, including describe how parts of graph are reflected in narrative (e.g., what's happening at the flat spaces, what's happening at spaces that aren't flat, what's happening at corners) and why narrative accounts for shape of graph • Understand how to interpret the shape of a qualitative graph in terms of co-varying quantities, including construct a story (narrative) to match a graph	situations based on recognizing constant rate of change • Construction of qualitative graph that accurately represents a story scenario • Construction of a story narrative that accurately conveys the shape of the graph of a function			
4.2) The multiple ways in which functions can be represented are connected mathematically in ways that allow for navigation among these representations.	• (4.2.a) Understand how to use a function table to construct a graph • (4.2.b) Understand how to use a graph to construct a	• (4.2.a.1) Inscriptions indicating transformation of data from function table into a graph	• Students have more difficulty translating from graph to equation than from equation to graph (Leinhardt et al., 1990, grades 3-9)	• Express in words and variables linear relationships presented verbally (Swafford &	

	corresponding function table • (4.2.c) Understand how to use a linear function of the form y=mx+b to construct a graph • (4.2.d) Understand how to use a linear function of the form y=mx+b to construct a function table • (4.2.e) Understand how to use a graph to identify slope and y-intercept and construct a function rule • (4.2.f) Understand connections between different representations of a function • (4.2.g) Understand that a constant rate of change depicted in a function table yields a linear graph or a function rule of the form y=mx+b (Connected Math Grade 7-Moving Straight Ahead) • (4.2.h) Understand that y-intercept of graph of linear function corresponds to point (0, b) in function table and the value b in the rule y=mx+b	(i.e., data from function table transformed into points and plotted on a graph) • (4.2.b.1) Inscriptions indicating points on a graph are identified and placed appropriately in a function table • (4.2.c.1) Inscriptions indicating parameters *m* and *b* are identified in function rule y=mx+b and used to identify y-intercept and slope of graph and corresponding graph constructed • (4.2.d.1) Inscriptions indicating substitution of appropriate values in function rule to determine corresponding pairs of function values, and their appropriate	• Students experience more success working within a given representation than translating among representations (Nathan et al., 2002, grades 7-8) • The processes of doing and un-doing (reversibility) are important precursors to the notion of inverse functions, a difficult idea for students • Reversals in thinking (e.g., given the number of tiles, tell the position number) can be difficult for students (Warren & Cooper, 2008, grade 3) • Students developed a notion of slope as a difference ("it goes up by") rather than a ratio as intended (Lobato, Ellis & Munoz, 2003, grade 9)	Langrall, 2000, grade 6) • Add linear graphs together, generate the final function and the original functions, and express the equivalence in symbols (Kieran & Sfard, 1999, grade 7) • Given a linear expression in symbols, a table, graph, and story constructed to represent it (Kieran & Sfard, 1999, grade 7) • Work within, create, and translate among verbal, graphical, tabular, and symbolic reps of linear functions (Brenner et al., 1997, grades 7-8)	

	• (4.2.i) Understand, for linear functions, connection between covariational relationship observed in table, the parameter m in the function rule y=mx+b (for a unit change in the independent variable, the change in the dependent variable is m), and the slope of the corresponding line • (4.2.j) Understand unit rate as slope of related line for proportional relationships y=mx • (4.2.k) Understand that the constant of proportionality in a proportional relationship y=kx is the slope, k, and the graph of this relationship is a line through the origin • (4.2.l) Understand that corresponding values in a function table must satisfy the function rule. That is, when function variables are substituted with corresponding values from the table, the result must be a true equation	placement in a function table • (4.2.e.1) Inscriptions indicating use of graph to identify slope and y-intercept and construction of a function rule using these data • (4.2.f.1) Reasoning illustrating use of different representations simultaneously to interpret a problem situation • (4.2.g.1) Descriptions indicating constant rate of change in function table yields a linear graph or function of form y=mx+b; construction of function rule of form y=mx+b or linear graph based on characterization of constant rate of change in function table • (4.2.h.1)			

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	• (4.2.m) Understand that a point of intersection of two graphs satisfies both function rules and appears in both function tables • (4.2.n) Understand differences between linear and nonlinear functions as depicted in function tables, graphs and rules • (4.2.o) Understand how to characterize connections between multiple function rules in a family of linear functions and related graphs (e.g., how does change in parameter value affect shape of graph; what is constant in family of functions (b, or y-intercept) and how is that reflected in the graphs) • (4.2.p) Understand connections between function rules for a system of linear equations (functions) and whether slopes (*m*) indicate lines are parallel, intersecting, or the same line	Description of y-intercept as point (0, b) or constant value, *b*, in y=mx+b • (4.2.i.1) Descriptions connecting covariational relationship in function table for linear function to the parameter *m* in y=mx+b and slope of line in graph • (4.2.j.1) Description of the connection between the unit rate in a problem situation and the slope of the related line • (4.2.k.1) Descriptions connecting constant of proportionality in linear function y=kx to slope of line and corresponding construction of linear graph through origin • (4.2.l.1) Written substitutions of specific pairs of			

	• (4.2.q) Understand how to evaluate function rules through substitution of corresponding data values to determine if the function rule is consistent with the data • (4.2.r) Understand how to use tables or function rules to make predictions about shape of graph	corresponding points in function rule, with computations indicating a true equation • (4.2.m.1) Written substitutions of point(s) of intersection(s) of two graphs in function rules, with computations indicating a true equation • (4.2.n.1) Explanations identifying functions as linear or nonlinear based on growth patterns in function table, shape of graph, or symbolic form of function rule • (4.2.o.1) Descriptions of connections between function rules for a family of linear functions and their graphical as representations related to the parameters *m* and *b*			

		• (4.2.p.1) Descriptions indicating whether slope values for functions in a system of linear equations reflect functions that represent intersecting or parallel lines or the same line • (4.2.q.1) Written substitutions that indicate correct substitution of corresponding values in a function rule • (4.2.r.1) Explanations of information in tables (e.g., growth patterns) or function rules (e.g., parameters in linear functions) and their implications for shape of graph			

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