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GENERALIZED ARITHMETIC

Generalized arithmetic involves generalizing arithmetic relationships, including properties of number and operation, and reasoning explicitly with these generalizations. In this context, generalized arithmetic entails reasoning about the structure of arithmetic expressions rather than their computational value.

STRAND 1. Relationships (Generalizing Relationships)

This strand identifies concepts related to relationships expressed in their generalized form in the context of GA.

CONSTRUCT (This column contains constructs within the Relationships strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
1.1) The Fundamental Properties of Number and Operation represent the underlying relationships that govern how operations behave and relate to each other: • Assoc prop of add • Assoc prop of mult • Comm prop of add • Comm prop of mult • Distributive prop • Zero is add identity	• Understand how to generalize Fundamental Properties • Understand how to identify fundamental properties in use in computations • Understand how to identify Fundamental Properties in use in transformations of algebraic expressions	• Conjectures, using words or variables, of Fundamental Properties • Identification of Fundamental Properties used in a computation • Identification of Fundamental Properties used in transformation of an algebraic expression	• Students may not believe that the properties hold for all real numbers • Students do not easily view algebra as "generalized arithmetic" and often believe numbers and variables behave differently (Lee & Wheeler, 1989, grade 10)	• Make generalizations when prompted by appropriate open or T/F number sentences (e.g., 58 + 0 = 58; 58 + 85 = 85 + 58) (Bastable & Schifter, 2008; Carpenter et al., 2003, grades 1-6)	• T/F and open number sentence to elicit generalizations (Carpenter et al., 2003, grades 1-6)

• One is mult identity • The additive inverse of a is –a • The multiplicative inverse of a is 1/a					
1.2) Operations are inversely related to each other: • Addition and subtraction have an inverse relationship • Multiplication and division have an inverse relationship	• Understand relationships among operations (e.g., division or repeated subtraction) and their inverse relationships (e.g., addition and subtraction have an inverse relationship)	• Statements about relationships among operations		• Relate operations in terms of doing and undoing (Driscoll, 1999) • Identify relationships among operations (Carpenter et al., 2003, grades 1-6; Schifter, 1999, grades 1-6; Slavit, 1999, grades 1&4)	• See Driscoll (1999, grades 6-10) • Ice cream (Schifter, 1999, grade 6)
1.3) Generalizations in arithmetic other than the Fundamental Properties can be derived from the Fundamental Properties. These include relationships in classes of numbers and outcomes of calculations (e.g., even and odd numbers; factors and divisibility; 'multiplication doesn't always make bigger' or '-x is not always a negative number').	• Understand how to generalize arithmetic relationships in contexts such as classes of numbers or outcomes of calculations	• Conjectures using words or variables about arithmetic relationships (e.g., about even and odd numbers, consecutive numbers and other number patterns	• Students may not grasp the full range of values variables or quasi-variables can take on (Fujii & Stephens, 2008)	• Make generalizations about properties of even and odd numbers (Bastable & Schifter, 2008, grade 1; Blanton, 2008; Carpenter et al., 2003; Kaput, 1998, 1999) • Make generalizations that derive from the fundamental properties by treating numbers as quasi-variables (e.g., 78 – 49 + 49 = 78) (Fujii & Stephens, 2008)	
1.4) The Fundamental Properties are relationships that are	• Understand why Fundamental Properties are true for	• Explanations that show the Fundamental	• Students may not believe that "always true" equations hold		

true for all values of the variables in a specified number domain.	all values of the variables in a specified number domain	Properties are true for all values of the variables.	for all numbers		

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STRAND 2. Representations (Representing Generalizations)

Generalized relationships can be represented in a variety of forms, including words, symbols, and pictures. This strand addresses representational knowledge for modeling, describing, or reasoning with these relationships in the context of GA.

CONSTRUCT (This column contains constructs within the Representations strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
2.1) Arithmetic generalizations can be described in words using natural language or with variables representing numbers in a generalized pattern.	• Understand how to express Fundamental Properties in words • Understand how to express Fundamental Properties using variables • Understand how to express arithmetic generalizations (other than Fundamental Properties) in words and, if appropriate, variables	• Conjectures of Fundamental Properties expressed in words • Conjectures of Fundamental Properties expressed in variables • Conjectures of arithmetic generalizations other than Fundamental Properties expressed in words or, if appropriate, variables.		• Express conjectures about even and odd numbers in words (Bastable & Schifter, 2008, grade 1; Blanton, 2008; Carpenter et al., 2003; Kaput, 1998, 1999) • Write conjectures about arithmetic properties symbolically (Carpenter et al., 2003) • Make "if…then" statements about the relationship between addition and subtraction (Carpenter et al., 2003)	

				• Express commutativity of addition and multiplication in words (Bastable & Schifter, 2008, grade 3) • Express relationship between consecutive square numbers in words (Bastable & Schifter, 2008, grade 4)	
2.2) Arithmetic generalizations can be expressed with generically-treated numbers.	• Understand how general arguments can be made using specific numbers if these numbers are treated generically, as quasi variables	• Use specific numbers as quasi variables in a general argument		• Use specific numbers in a general way when using physical models (e.g., unifix cubes) to justify fundamental properties (Russell, Schifter, & Bastable, 2011)	
2.3) Arithmetic generalizations can be represented in pictorial forms of a general nature.	• Understand that while pictorial representations of arithmetic generalizations may use specific numbers, general arguments can be drawn from them	• Create and interpret pictorial representations of arithmetic generalizations		• Create pictorial representations of arithmetic generalizations (Russell, Schifter, & Bastable, 2011)	

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STRAND 3. Justifications (Justifying Generalizations)

Generalized relationships can be justified[1] using different strategies. This strand describes how justification occurs in the context of GA.

CONSTRUCT (This column contains constructs within the Justification strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH-BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
3.1) Arithmetic generalizations (including Fundamental Properties) are sometimes justified empirically, by substituting specific numbers in the generalization.	• Understand how to develop empirical arguments (e.g., testing examples) to examine conjectures about arithmetic relationships • Understand limitations of empirical arguments	• Arguments supporting or refuting conjectures about arithmetic relationships that are based on testing specific numerical cases • Explanations that show limitations of empirical arguments (e.g., recognition that not all numbers can be tested)	• Students may believe that conjectures can be justified by example (Knuth, 2002)	• Justify properties of evens and odds (Bastable & Schifter, 2008, grade 1; Blanton, 2008; Carpenter et al., 2003; Kaput, 1998, 1999)	
3.2) Arithmetic generalizations are sometimes justified by an algebraic use of numbers, where arguments use specific cases in a way that does not depend on specific	• Understand how to use numbers algebraically to justify conjectures of the Fundamental Properties • Understand how to use numbers algebraically to justify conjectures	• Arguments supporting conjectures about arithmetic relationships that use numbers algebraically (i.e., as quasi-variables)		• Use specific numbers in general arguments supporting fundmental properties (Russell, Schifter, & Bastable, 2011)	

[1] The idea of justification (rather than "proof") is used here intentionally. Essentially, these refer to mathematical arguments for why a particular conjectured relationship is a reasonable conjecture.

cases or numbers (Carpenter, Franke & Levi 2003).	about arithmetic generalizations other than the Fundamental Properties				
3.3) Arithmetic generalizations are sometimes justified by reasoning with previously established generalizations.	• Understand how to use established generalizations to justify conjectures about arithmetic relationships	• Written or oral arguments supporting conjectures about arithmetic relationships that invoke previously established generalizations		• Justify generalization by reasoning with previously-established ones (Carpenter et al., 2003)	
3.4) Arithmetic generalizations are sometimes justified by using representation-based reasoning (see Schifter 2008).	• Understand how to construct and use representations to justify conjectures about arithmetic relationships	• Justifications of conjectures about arithmetic relationships using representation-based reasoning		• Justify conjectures about arithmetic relationships using representation-based reasoning (Bastable & Schifter, 2008; Carpenter et al., 2003; Schifter et al., 2008)	
3.5) Arithmetic generalizations are sometimes justified by using algebraic arguments based on transforming algebraic expressions.	• Understand how to construct algebraic arguments as appropriate to justify arithmetic generalizations	• Algebraic argument that uses a chain of logic to establish an arithmetic generalization • Justify steps in solving an algebraic equation (e.g., explain the decision to add 5 to both sides)			
3.6) Justifications based on general arguments show that a conjecture	• Understand why general arguments show that a conjecture is true	• Critiques of justifications based on non-general	• Students have difficulty seeing how algebra can be used to	• Make general arguments to justify conjectures about	

is true for all possible cases (e.g., numbers in a specified number domain) and thus represent the strongest possible argument.	for all numbers in a specified number domain	arguments that indicate limitations of the justification • Justifications based on general arguments and descriptions of affordances of general argument	justify a general statement about numbers (Kieran, 2007)	arithmetic properties (Carpenter et al., 2003, grades 1-6) • Justify conjectures about even and odd numbers in words (Blanton, 2008; Carpenter et al., 2003) • Justify use of the associative and distributive properties to simplify calculations using words and/or manipulatives (Carpenter et al., 2003)	
3.7) While general arguments are needed to prove a conjecture is true, only one counterexample is needed to prove a conjecture false.	• Understand why general arguments are needed to prove conjectures true, but only one counterexample is needed to prove a conjecture false	• Evaluate appropriateness of using examples to prove true/false conjectures • Use counterexamples to prove a conjecture is false (e.g., a – b = b – a)			

STRAND 4. Reasoning with Generalizations
This strand describes putting generalizations about arithmetic relationships "to use" in reflecting on new generalizations or in performing computational tasks.

CONSTRUCT (This column contains constructs within the Reasoning strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH-BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
4.1) Generalizations in arithmetic can be used as objects for reasoning about or with conjectured arithmetic relationships.	• Understand how to reason with arithmetic generalizations expressed in words or variables	• Arguments or explanations that invoke previously established generalizations when reflecting on a given generalization			
4.2) Fundamental properties can be used to compose or decompose quantities in insightful ways	• Understand how to simplify calculations by use of the fundamental properties (e.g., 98+ 137 + 2 = 98 + 2 + 137) • Understand how to use numbers as quasi-variables to simplify calculations	• Use of Fundamental Properties to simplify calculations • Use of numbers as quasi variables when using Fundamental Properties to simplify computations • Solutions to equations based on "undoing" or	• Students may have difficulty identifying the properties they use when they transform algebraic expressions (Kieran, 2007) • Students may misapply fundamental properties (e.g., 2(xy) = (2x)(2y)) (Russell, Schifter & Bastable, 2011)	• Make use of the associative and distributive properties to simplify calculations (Carpenter et al., 2003) • Use relational thinking and arithmetic properties to solve problems (Jacobs et al., 2007; Koehler, 2004, grades 2-3;	

		reversing operations • Identification of fundamental properties when simplifying expressions or solving equations		Schifter, 1999, grades 1-6) • Compose and decompose terms to lead to canceling strategies in solving some linear equations (Linchevski & Herscovics, 1996, grade 7) • View numbers as quasi variables when using arithmetic properties to simplify computations (Irwin & Britt, 2005, grade 8) • Use the associative and distributive properties (i.e., partitioning strategy) to solve multiplication problems (Baek, 2008, grades 3-5)	
4.3) Knowledge about relationships between arithmetic operations can be used to solve problems in multiple ways	• Understand how relationships between operations can be used to solve problems in multiple ways • Understand how to use unwinding strategies that make use of the	• Critique multiple solutions to the same problem and explain why they are all valid by discussing relationships between operations		• Demonstrate relationship between division and repeated subtraction (Schifter, 1999, grade 6) • Demonstrate	

	relationships between operations	• Solve problem in more than one way, using knowledge of relationship between operations (e.g., 45 – 32 can be solved by subtracting or counting up) • Written or oral mathematical work that shows reversing operations (e.g., in solving an equation) • Multiple solutions to equations based on relationships among operations (e.g., division as repeated subtraction)		relationship between addition and subtraction (Bastable & Schifter, 2008, grade 2; Schifter, 1999, grades 1-2) • Model division of whole number by fraction using pictures and represent using variety of number sentences (Schifter, 1999, grade 6)	

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