PROPORTIONAL REASONING

When two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same numerical factor.

Strand 1. Generalizing relationships
Relationships between two co-varying quantities are proportional if the ratio of one quantity to another remains constant. This strand identifies content related to proportional relationships that might be generalized.

CONSTRUCT (This column contains constructs within the Relationships strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
1.1) A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change by the same numerical factor.	• Understand that a proportion is a relationship of equality between two ratios and that while individual quantities may change by the same factor, their ratios remain constant	• Identification of whether or not a relationship (described verbally or with data points) is a proportional one			
1.2) A ratio can act as a measure of a particular attribute	• Understand that a ratio can act as a measure (e.g., slope as a measure of steepness)	• Identification of what attribute a particular ratio is measuring	• Students have difficulty seeing slope as a measure or articulating its meaning (Lobato & Thanheiser, 2002, grade 9)		• Steepness of ramp (Lobato et al., 2010, grades 6-8)
1.3) A rate is a set of infinitely many equivalent ratios. These ratios may	• Understand the idea of an infinite class of equivalent ratios	• Statements about the fact that there are infinitely many ratios equivalent to any			

involve comparison of quantities with the same or different units.		given ratio			
1.4) Direct proportions are proportions where one quantity increases (or decreases) as another quantity increases (or decreases) , and their ratio remains constant.	• Understand whether situations involving proportionality involve direct relationships	• Identification of whether direct relationships exist in situations involving proportionality • Examples of relationships that are directly proportional			
1.5) Inverse proportions are proportions where one quantity decreases as the other increases, and their product remains constant.	• Understand whether situations involving proportionality involve inverse relationships	• Identification of whether inverse relationships exist in situations involving proportionality • Examples of relationships that are inversely proportional			• Gear task (teeth to rotations is inverse relationship) (Ellis, 2007)

Strand 2. Expressing generalizations

Proportional relationships can be represented in a variety of forms, including words, symbols, tables and graphs. This strand addresses types of representations that are used to model, describe or reason with proportional relationships.

CONSTRUCT (This column contains constructs within the Representations strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
2.1) The relationship between two quantities, x and y, that vary proportionally can be described by the function y = mx, or a table in which a unit change in x results in a change of magnitude m in y, where m is the constant of proportionality.	• Understand the meaning of the constant of proportionality and how it is represented in an equation or table in relationships described by y = mx	• Expression of the meaning of the constant of proportionality • Identification of the constant of proportionality given a function rule or table of the form y = mx			
2.2) Graphs of pairs of quantities that are directly proportional are lines that pass through the origin, with the slope equal to the constant of proportionality.	• Understand the constant of proportionality in the context of a graph of the form y = mx • Understand that graphs of lines through the origin represent directly proportional relationships	• Identification of graphs of lines that pass through the origin as representing directly proportional relationships • Identification of the constant of proportionality given a graph of the form y = mx • Graphs of directly proportional relationships			

		• Explanation of why directly proportional relationships produce graphs that are lines through the origin			
2.3) The relationship between two quantities, x and y, in which change in y is proportional to change in x but the quantities themselves are not proportional, can be described by the function y = mx + b or a table in which a unit change in x results in a change of magnitude m in y (with y ≠ 0 when x = 0), where m is the constant of proportionality.	• Understand the meaning of the constant of proportionality and how it is represented in an equation or table in relationships described by y = mx + b	• Expression of the meaning of the constant of proportionality • Identification of the constant of proportionality given a function rule or table of the form y = mx + b			
2.4) Relationships in which the change in one quantity is proportional to the change in the other but the quantities themselves are not proportional are depicted by graphs of lines with y-intercepts not equal to zero.	• Understand the constant of proportionality in the context of a graph of the form y = mx + b • Understand that graphs of lines not passing through the origin represent relationships in which the change in one quantity is proportional to the change in the other but the quantities themselves are not proportional • Understand the	• Identification of graphs of lines that do not pass through the origin as representing relationships in which the change in one quantity is proportional to the change in the other but the quantities themselves are not proportional • Identification of the constant of proportionality given a graph of the form y = mx + b			

	difference between graphical representations of proportional relationships (y = mx) and relationships in which the change in quantities is proportional but the quantities themselves are not (y = mx + b)	• Graphs of relationships in which the change in one quantity is proportional to the change in the other but the quantities themselves are not proportional • Explanation of why relationships in which the change in one quantity is proportional to the change in the other but the quantities themselves are not proportional produce graphs that are lines through the origin • Explanation of the difference between the graphs of functions y = mx and y = mx + b			
2.5) Graphs of inversely proportional relationships are hyperbolas	• Understand that hyperbolas represent inversely proportional relationships				

Strand 3. Justifying generalizations

Generalized relationships in proportional data can be justified[1] using different strategies. This strand describes how justification occurs in the context of proportional reasoning.

CONSTRUCT (This column contains constructs within the Justification strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
3.1) The existence of a directly proportional relationship, the existence of a relationship in which the change in one quantity is proportional to the change in the other but the quantities themselves are not proportional, or the existence of an inversely proportional relationship can be verified by referring to a verbal description of the situation, a table, a graph, or an equation	• Understand how to justify the existence of a directly proportional relationship, a relationship in which the change in one quantity is proportional to the change in the other but the quantities themselves are not proportional, or an inversely proportional relationship from given data	• Justify the existence of a directly proportional relationship by referencing a verbal description of the relationship, a table, a graph, or an equation • Justify the existence of a relationship in which the change in one quantity is proportional to the change in the other but the quantities themselves are not proportional by referencing a verbal description of the relationship, a table, a graph, or an equation			

[1] The idea of justification (rather than 'proof') is used here intentionally. Essentially, these refer to mathematical arguments for why a particular conjectured relationship (in this case, a proportional relationship) is a reasonable conjecture.

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		• Justify the existence of an inversely proportional relationship by referencing a verbal description of the situation			

Strand 4. Reasoning explicitly with generalizations

Reasoning with generalizations includes both actions on and actions between representations. These actions might be based on manipulation of symbolic forms using syntactical rules or reasoning with other representations such graphs or (ratio) tables to understand proportional relationships.

CONSTRUCT (This column contains constructs within the Reasoning strand.)	**CLAIMS (UNDERSTANDINGS)** (This column contains understandings that students are expected to eventually acquire over the grades 3-8 progression. They will have partial understandings and misconceptions along the way towards these understandings.)	**EVIDENCE** (This column contains what students' oral or written work or activity should include.)	**DIFFICULTIES & MISCONCEPTIONS** (This column contains common misconceptions or difficulties revealed by research.)	**RESEARCH BASE** (This column contains what research tells us students can do.)	**TASKS** (This column contains research-based tasks.)
4.1) Proportional reasoning involves understanding that— • Equivalent ratios can be created by iterating and/or partitioning a composed unit • If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship	• Understand how to maintain proportional relationships, build units, and iterate to find missing values	• Building of composite units and iteration to solve proportion problems	• Students may use an incorrect build-up strategy: multiplicative strategy on non-integer problem and then using incorrect additive thinking to handle remainder (Misailidou & Williams, 2003, grades 5-8) • Students may double or halve when working with proportions when this is not appropriate (Misailidou & Williams, 2003, grades 5-8) • Students may believe the sum of each person's items in a	• Solve proportion problems using unit rate approach (Christou & Phillippou, 2002, grades 4-5), (Lo & Watanabe, 1997, grade 5), (Kaput & West, 1994, grade 6; Singh, 2000, grade 6) • Begin thinking multiplicatively in grade 2, develops throughout elementary grades (Clark & Kamii, 1996, grades 1-5) • Use "building up" strategy to solve proportion	• Pizza task (Lamon, 2007, p. 644) • Cookie task (Lamon, 1999, p. 96) • Eraser task (Lamon, 1999, p. 100) • Vehicle task (Lamon, 1999, p. 101) • Big dinner (Fosnot, 2007) • 15 tasks in Kaput & West (1994, grade 6) • Coffee task (Clark et al., 2003) • Map scale (Weinberg, 2002) • Crackers (Lobato

			proportion situation should be equal (Misailidou & Williams, 2003, grades 5-8) • Features that make proportion problems easier include reduced forms of ratios, familiar multiples, containment (two sets of items held together), "for each/every statements, and familiar rates (e.g.,speed and price) (Kaput & West, 1994, grade 6) • Features that make proportion problems difficult include two quantities not dividing evenly and small difference between quantities in a ratio (the latter leading to the incorrect additive strategy) (Kaput & West, 1994, grade 6) • Geometric shape proportion problems are among the most difficult and often lead to the incorrect additive strategy (Kaput & West, 1994, grade 6)	problems (Lo & Watanabe, 1997, grade 5), (Kaput & West, 1994; Lamon, 1994; Singh, 2000, grade 6)	et al., 2010, grades 6-8) • See Misailidou & Williams (2003) appendix for validated items (grades 5-8) • Cat food (Fosnot, 2007)

4.2) Forming a ratio as a measure of a real-world attribute involves isolating that attribute from other attributes and understanding the effect of changing each quantity on the attribute of interest.	• Understand that forming a ratio as a measure of a real-world attribute requires isolating that attribute from others • Understand how changing each quantity in a ratio impacts the ratio (and the attribute it is measuring) as a whole	• Reasoning about how changing individual quantities in a ratio impacts the ratio as a whole	• Students sometimes have difficulty determining the effect of changing one quantity at a time on the ratio (Lobato, 2008, grade 9)		
4.3) There are infinitely many ratios that can be identified that are equivalent to a given ratio	• Understand how to generate a list of equivalent ratios	• Generation of a list of many equivalent ratios, including those involving decimals or fractions that go beyond "easy" ratios (e.g., those found by doubling)	• Students may be able to find equivalent ratios through doubling and other "easy" strategies but may have difficulty finding those with messier numbers (Lobato et al., 2010, grades 6-8)		• SimCalc frog/clown generate equivalent ratios (Ellis, 2007, grade 6)

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