FROM RECURSIVE PATTERN TO CORRESPONDENCE RULE: DEVELOPING STUDENTS' ABILITIES TO ENGAGE IN FUNCTIONAL THINKING

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Third- through fifth-grade students participating in a classroom teaching experiment investigating the impact of an Early Algebra Learning Progression completed pre- and post-assessment items addressing their abilities to engage in functional thinking. We found that after a sustained early algebra intervention, students grew in their abilities to shift from recursive to covariational thinking about linear functions and to represent correspondence rules in both words and variables.

Keywords: Algebra and Algebraic Thinking, Elementary School Education

Algebra has historically served as a gateway course to higher mathematics that—due to high failure rates—has been closed for many students. More recent initiatives have identified algebra as playing a central role throughout mathematics education and re-framed it as a longitudinal strand of thinking across grades K-12 rather than as an isolated eighth- or ninth-grade topic (e.g., Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000). This is not to be interpreted as a call to shift traditional algebra instruction to earlier grades, but rather as one to introduce elementary school students to algebraic thinking in the context of age-appropriate activities.

In response to this call, we drew from research findings, curricular resources, and standards documents in the area of early algebra to develop an Early Algebra Learning Progression [EALP] organized around five "big ideas": 1) Generalized Arithmetic, 2) Equations, Expressions, Equality, and Inequality, 3) Functional Thinking, 4) Proportional Reasoning, and 5) Variable.

We conducted a one-year classroom-based study in grades 3-5 to gather efficacy data regarding the impact of EALP-based classroom experiences on elementary students' developing understandings of these big ideas. The focus of this paper will be our findings regarding the development of students' functional thinking. We will share pre/post assessment data and representative excerpts from student work and briefly discuss the classroom intervention we believe contributed to the growth we observed.

Theoretical Perspective

Functional thinking has been identified as one of the key strands of algebraic thinking (Kaput, 2008) and one of the core content domains in early algebra research (Blanton, Levi, Crites, & Dougherty, 2011). Blanton et al. (2011) characterize functional thinking as "generalizing relationships between covarying quantities, expressing those relationships in words, symbols, tables, or graphs, and reasoning with these various representations to analyze function behavior" (p. 47).

Elementary curricula often include a focus on simple patterning activities in which the change in only one variable is observed. However, an exclusive focus on this type of activity is suggested to hinder the development of students' reasoning about how two or more quantities vary simultaneously (Blanton & Kaput, 2004). A deeper understanding of change and the modeling of behavior in real world phenomena requires students to look beyond recursive patterns and consider covariation in their study of functions (Blanton et al., 2011). Confrey and Smith (1994) furthermore argue that a covariational approach establishes a good foundation for the development of correspondence rules.

Evidence exists that children in elementary grades are in fact capable of reasoning about covarying quantities and developing correspondence rules. Blanton and Kaput (2004) found that students could construct and reason with function tables and identify covariational relationships and primitive correspondence rules as early as first grade, while older elementary students could successfully transition from using natural language to using symbols to represent correspondence rules more formally. Based on these findings, Blanton and Kaput argue that elementary curricula should go beyond recursive pattern finding to include a focus on relationships between variables.

Martinez and Brizuela (2006) likewise found that third-grade students could successfully reason with linear function tables but sometimes struggled to make the transition from focusing on recursive patterns to identifying general correspondence rules that would apply to all cases. They identified "hybrid" approaches, in which students examined the relationship between input and output as required in a covariational approach while simultaneously relying on a recursive pattern. For example, one student observed in her table that "the number that you add to get from the input to the output is always one more than it was in the previous row" (p. 292). This is a limited approach in terms of generalizing and making far predictions, but it does indicate progress in considering the relationship between two variables.

The study of functions in the elementary grades can lay the foundation for success in later grades. Teachers can nurture students' functional thinking by helping them develop algebraic habits of mind that encourage building patterns, making conjectures, generalizing, and justifying mathematical relationships (Blanton & Kaput, 2011; Moss, Beatty, Barkin, & Shillolo, 2008; Moss & McNab, 2010). Martinez and Brizuela (2006) call for carefully designed interventions that consider the relationship between "what students *know* and what we want them to *learn"* (p. 293). In this study, we aimed to move beyond the patterning experiences elementary curricula and standards documents (e.g., National Council of Teachers of Mathematics, 2000) propose students should have and push students to consider covariational relationships and develop correspondence rules. Specifically, this paper addresses the following research question:

How does the functional thinking of grades 3-5 students who have had a year-long focus on early algebra (including functions) compare to that of students who have had more traditional arithmetic-based experiences? Specifically, how does student performance compare across the following aspects of functional thinking:

a) Constructing a function table?
b) Identifying a recursive pattern?
c) Identifying a covariational relationship?
d) Representing a correspondence rule in words and symbols?
e) Making "far" data predictions?

Method

Participants

Participants included approximately 300 students from two elementary schools in southeastern Massachusetts. The school district in which these schools reside is largely white (91%) and middle class, with 17% of students qualifying for free or reduced lunch. Six classrooms (two from each of grades 3-5 and all from one school) served as experimental sites and 10 classrooms (four grade 3, four grade 4, and two grade 5, from both schools) served as control sites.

Classroom Intervention

Students in the experimental condition participated in an EALP-based classroom teaching experiment [CTE] for approximately one hour each week for one school year. A member of our research team—a former elementary school teacher—served as the teacher during these interventions. A typical one-hour lesson consisted of a "jumpstart" at the beginning of class to review previously-discussed concepts, followed by group work centered on research-based tasks aligned with our EALP. These tasks were designed to encourage students to reason algebraically in a variety of ways and justify their thinking to themselves and their classmates.

The last five weeks of the CTE focused on functional thinking, in particular, problem situations in which students investigated linear patterns and relationships. In most of these tasks, students were presented with a scenario in words or pictures and were asked to record and organize data, identify and describe recursive patterns and covariational relationships, express correspondence rules in words and symbols, and make near and far predictions. Multiple representations—verbal, pictorial, tabular, graphical, and symbolic—were typically generated from a given problem context. Students were encouraged to discuss connections among representations (e.g., to identify the meaning of the slope and intercept in a symbolic correspondence rule by referring to the function table or by referring back to the original problem context).

Students in the control condition participated in their usual classroom activity with their regular classroom teachers. District-wide, all classroom teachers used "Growing with Mathematics" (Iron, 2003) curriculum materials. This curriculum does not include a focus on early algebra.

Data Collection

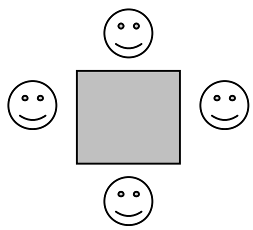
A pretest and (identical) posttest were designed to measure students' understandings of algebraic topics identified across the five "big ideas" of the EALP. The majority of tasks were research-based, adapted from tasks used in our or others' prior studies. In total, 290 students completed the pretest (117 experimental, 173 control) and 293 students completed the posttest (126 experimental, 167 control). We also conducted individual interviews with ten students (6 experimental, 4 control) across grades 3-5 at the conclusion of the study to gain deeper insight into their thinking about a subset of the assessment tasks.[1]

From the pre/post assessment, we will focus in this paper on one task—the *Brady task* (see Figure 1)—that investigated students' functional thinking around a situation involving linear growth.

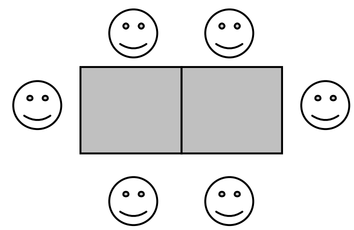
[1] Due to limited space, we do not discuss interviews here but will share representative excerpts in our presentation.

Brady is having his friends over for a birthday party. He wants to make sure he has a seat for everyone. He has square tables.

He can seat 4 people at one square table in the following way:



If he joins another square table to the first one, he can seat 6 people:



a) If Brady keeps joining square tables in this way, how many people can sit at 3 tables? 4 tables? 5 tables? Record your responses in the table below and fill in any missing information:

Number of tables	Number of people
1	
2	
3	
4	
5	
6	
7	

b) Do you see any patterns in the table? Describe them.

c) Find a relationship between the number of tables and the number of people who can sit at the tables. Describe your relationship in words.

d) Describe your relationship using variables. What do your variables represent?

e) If Brady has 10 tables, how many people can he seat? Show how you got your answer.

Figure 1: The *Brady task*

Data analysis

Each part of the *Brady task* was first scored dichotomously (i.e., correct or incorrect). For all but part a (which required no explanation), student strategies were also coded.

For parts b, c, and d, student responses were categorized according to the type of relationship described: *recursive*, *covariational*, or *functional*. For example, the most prevalent response to part b was for students to provide a description of a recursive pattern (e.g., “The people column goes up by 2s.”). We anticipated students would respond in this way, given the focus of typical elementary curricula, and thus designed parts c and d to try to push students beyond recursive thinking. Part c required students to consider the relationship between two variables, thus requiring either a description of a covariational relationship (e.g., "When the number of tables goes up by 1, the numbers of people goes up by 2.") or a correspondence relationship (e.g., "The number of people is 2 more than 2 times the number of tables."). In part d, students were expected to describe the correspondence relationship symbolically (e.g., "$2n + 2 = p$ where n = number of tables and p = number of people.").

Student responses to part e were coded according to the strategy used to determine the number of people who could sit at 10 tables: *drawing* indicated that students drew 10 tables and counted the number of people who could be seated, *recursive* indicated that students extended the pattern found in the table in part a to 10 tables, and *functional* indicated that students used the correspondence relationship between the two variables to find the solution (i.e., $2 \times 10 + 2 = 22$ people). Student responses to part e that included no work or explanation were placed into an *answer only* category.

Across all of the items, responses that students left blank, or for which they responded "I don't know" were grouped into a *no response* category, and responses that were not sufficiently frequent to constitute their own codes were placed into an *other* category.

To assess reliability of the coding procedure, a second member of the research team coded a randomly selected 20% sample of the data. Initial agreement between coders was at least 74% for each item. All differences in scoring were discussed by the coders and resolved.

Results and Discussion

In this section, we report pre/post results from the *Brady task* and offer representative excerpts from the written assessment to illustrate particular categories of responses.

Completing a table (part a)

In the first part of the *Brady task*, students were asked to complete a function table using the given description of the problem situation and accompanying pictures. Third- and fourth-grade students struggled with this task at pretest (see Table 1), while fifth-grade students were already fairly successful prior to the intervention. Grades 3-4 experimental students made significant improvements over the course of the intervention and outperformed control students at posttest.

Table 1: Proportion of students who successfully completed the table in response to part a

	Grade 3		Grade 4		Grade 5	
	Pre	Post	Pre	Post	Pre	Post
Control	.379	.524	.449	.716	.816	.946
Experimental	.359	.868*	.512	.932*	.857	.955

*Experimental group outperformed control group at posttest ($p < 0.01$).

These findings are consistent with Blanton and Kaput's (2004) and Martinez and Brizuela's (2006) assertions that provided the appropriate experiences, elementary students can learn to construct function tables to represent covarying data. While those students with an arithmetic-based curriculum could successfully construct tables by fifth grade, those students with early algebra experiences could do so sooner.

Recognizing and describing a pattern (part b)

Students were next asked to identify any patterns they saw in the table. This is a task with which we expected students to be fairly successful as only the identification of a recursive pattern was required. One third-grade student stated at pretest, for example, "You count by 2's every time." Most students took this recursive approach; however, some students identified a covariational relationship. A fourth-grade student, for example, wrote "plus 1 table = plus 2 more people" at pretest, indicating attention to the relationship between two variables. One fifth-grade student in the experimental group wrote "× 2 + 2" at posttest, indicating he or she was attending to the functional relationship between the number of tables and the number of people. Table 2 shows the proportion of students who provided a correct pattern or relationship to describe the data in the table. Overall posttest differences were only marginally significant at grade 3.

Table 2: Proportion of students who provided a correct recursive, covariational or functional table description in response to part b

	Grade 3				Grade 4				Grade 5			
	Control		Experimental		Control		Experimental		Control		Experimental	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Recursive	.182	.333	.205	.579	.275	.418	.415	.341	.579	.541	.476	.386
Covariational	.015	.079	.103	.211	.130	.254	.098	.296	.237	.351	.214	.409
Functional	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000	.023
Total correct	.197	.412	.308	.790*	.405	.672	.513	.637	.816	.892	.690	.818

*Experimental group outperformed control group at posttest ($p < 0.01$).

That students had some initial success with this task—especially by grade 5—is not surprising given the fact that elementary curricula typically focus on identifying recursive patterns in their work with number sequences and data tables. It is interesting to note, however, that an increasing proportion of students provided descriptions of the covariational relationship involved. Note that by fifth grade, more students in the experimental group were providing such responses than were identifying recursive patterns, suggesting the CTE successfully encouraged them to consider relationships between variables.

Expressing a functional relationship using words and variables (parts c and d)

Students were next explicitly asked to move beyond recursive patterning to consider the covariational or functional relationship between the number of tables and the number of people. We initially anticipated this would be very difficult for students given the lack of focus on these concepts in typical elementary curricula. Table 3 shows the proportion of students who provided a correct description of the covariational or functional relationship in words (part c) and the functional relationship in symbols (part d).

Table 3: Proportion of students who provided a correct covariational or functional relationship in words (in response to part c) and symbols (in response to part d)

	Grade 3				Grade 4				Grade 5			
	Control		Experimental		Control		Experimental		Control		Experimental	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Relationship in words (part c)												
Covariational	.015	.079	.026	.237	.101	.164	.146	.136	.316	.487	.119	.182
Functional	.000	.000	.000	.079	.000	.000	.000	.273	.026	.027	.000	.341
Total correct	.015	.079	.026	.316*	.101	.164	.146	.409*	.342	.514	.119	.523
Relationship in symbols (part d)												
Functional	.000	.000	.000	.158*	.000	.000	.000	.295*	.000	.000	.000	.455*

*Experimental group outperformed control group at posttest ($p < 0.01$).

As Table 3 shows, students struggled with these tasks at pretest. Only one student wrote a correct correspondence (i.e., functional) rule in words at that time. All other correct responses to part c at pretest involved describing in words the covariational relationship between the number of tables and the number of people. A fifth-grade student wrote, for example, “The rule is if you add a table two more people can sit.” No students wrote a correct symbolic functional rule in response to part d at pretest. Experimental students improved in this area quite a bit over the course of the intervention, with over 30% of fourth graders and almost half of fifth graders producing correct symbolic rules at posttest. For example,

$A \times 2 + 2 = B$; A for the number of tables, B for the number of people (grade 3)

$x \cdot 2 + 2 = y$; x represents the number of tables, y represents the number of people who sit at the tables (grade 4)

$p \times 2 + 2 = m$; p = # of tables, m = # of people (grade 5)

We attribute this performance to experimental students’ ongoing experience working with variables in a variety of contexts and to the connections continuously made among various representations and the original problem context in the CTE.

Making a “far” prediction (part e)

Finally, students were asked how many people Brady could seat at his party if he had ten tables. As described in the data analysis section, students took three main approaches: drawing ten tables and counting the number of people who could be seated, extending the pattern found in the table in part a to ten tables, or using the functional relationship between the two variables. See Table 4 for the proportion of students who correctly used each approach. “Answer only” refers to students who only answered “22,” with no work shown to indicate strategy use.

Table 4: Proportion of students who correctly applied a drawing, recursive, functional, or “answer only” strategy to make a “far” prediction in response to part e

	Grade 3				Grade 4				Grade 5			
	Con		Exp		Con		Exp		Con		Exp	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Drawing	.167	.143	.077	.132	.203	.284	.220	.296	.368	.324	.333	.091
Recursive	.076	.191	.128	.237	.188	.224	.220	.205	.237	.487	.286	.296
Functional	.000	.000	.000	.079	.000	.000	.024	.273	.026	.027	.024	.364
Answer only	.030	.079	.128	.105	.073	.119	.073	.000	.053	.054	.048	.068
Other	.000	.000	.000	.000	.000	.045	.000	.023	.000	.054	.000	.046
Total correct	.243	.413	.333	.553	.464	.672	.537	.797	.684	.946	.691	.865

Students in both control and experimental conditions showed improvement with this task, but there were no significant posttest differences between groups in terms of correctness. This is not entirely surprising given that the “far” prediction—to 10 tables—is not actually that far. Thus drawing and recursive strategies are not all that inefficient. In subsequent administrations of this task, we plan to ask students how many people could sit at 100 tables. Note, however, the experimental group’s increasing use of a function rule to help them solve this problem. We again attribute this difference to the CTE's focus on moving beyond recursive patterning to consider covariational relationships and correspondence rules.

Conclusion

Experimental students showed significant improvement in this study in their abilities to construct function tables (in grades 3 and 4), identify patterns or relationships in tables (in grade 3), and represent a functional rule verbally (in grades 3 and 4) and symbolically (in all grades). These findings support the work of others (e.g., Blanton & Kaput, 2004; Martinez & Brizuela, 2006) who assert that elementary students are capable of sophisticated functional thinking and call into question the lack of focus on relationships between co-varying quantities in many elementary curricula and recent standards documents.

Endnote

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